Foundations of the Empiricist Theory of Sets and Set Functions and its Logic

著者	KINOKUNIYA Yoshio
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Yoshio Kinokuniya*

Abstract

In Part I, previous results on the theory of a priori measure are rectified, renovated and rearranged for a systematic course of lectures. Herewith, it is specially emphasized that a euclidian space may be thought as a model of phenomenal field of physical events, independently of any metamathematical view on set theories. Besides, empiricism is thought to be essential to our inferences. In Part II, logical investigations are shown, standing on the empiricist view, and the principle of trans-induction is brought forward in a renovated form.

Introduction

Sets in a euclidian space may be taken up as the first and fundamental objects in empiricism. But the notion of a single point will then be nonsensical if shown independently of the space in which it dwells, because a 'point' must lose its actual sight of existence if it accompanies nothing to build its spatial neighborhood arround it. In this view, the 'space' may appear to be antecedent to a 'point'. On the other hand, the euclidian space has been used as a model of the phenomenal field of physical events, directly connected to our intuition, from the ancient days of Euclidus. In fine, geometrical forms in this space comprehend many meanings, historically accumulated through experiments and investigations, which had been made before the set theory was started. These being so, the set theory shall restrain itself from spoiling any aspect of the above-mentioned historical knowledges, which shall positively be qualified as the guides for correction over all of the theories connected to the euclidian space. Standing on this view the theory of a priori measure $\widetilde{m}$ was constructed. While some amount of works on the measure $\widetilde{m}$ were made by the present author, some occasional changes or alternations thereof could not be helped. In Part I, an ultimate coures of lectures is tried to settle some problems on $\widetilde{m}$ and to give some preliminary foundations for forthcoming studies of set functions.

Among the recent works on foundations of mathematics, the influence of symbolic logic may be marked as a conspicuous vogue. However, if symbolic logic be simply applied with empiricism, it is feared that the universe of objects

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* 紀国谷芳雄

may then be obscured by some metamathematical mist of abstraction. In effect, some investigation on the relation of 'implication' to 'logical range' have discovered a possible discrepancy between concrete objectification and abstract one. Moreover, empiricism prohibits the use of transfinite ordinals of higher class than the 3rd, so that the transfinite induction cannot be applied here. Besides, in fact, the transfinite treatment beyond the 3rd class, is essentially discordant with the a priori measure in a euclidian space. The logical investigations on the theory of a priori measure in empiricism are shown in outline in Part II.

I. Sets and Measures in a Euclidian Space

1. A Priori Measure

Length, area and volume may be cited as geometric events which from ancient times have been evident to human intuition. These events are namely geometric figures, and are equally called sets of points by the recent terminology. They are thereby defined as the measures of a set in one, two and three dimensions respectively. We specially call it *a priori measure* in the meaning that it is essential to human intuition. The a priori measure of a set M is written as $\widetilde{m}M$ (or $\widetilde{m}(M)$). $\widetilde{m}M$ is then the numerical value which indicates the largeness of the space occupied by the set M. In this case, the set M is considered to be contained in a euclidian space. However, we extend and generalize the space a little far, and by $\boldsymbol{E}$ we mean a general finite dimensional euclidian space.

In case of dimension 1, we have

$$\widetilde{m}\,[0,\ a] = a$$

$[0,\ a]$ being the closed interval $\{x:\ 0 \leqslant x \leqslant a\}$. As a increases, the part occupied by $[0,\ a]$ increases. This occupation is thought to be realized by points contained in $[0,\ a]$. Since a point, however, was defined as an interval which has only its position in the space and no largeness to be counted, it has been thought difficult to construct the measure of a set by means of the points contained in it. When we let a point x correspond to the point λx ($\lambda > 0$), we may naturally suppose that the size of a point λx should be given by multiplication of the size of the point x by λ, so that we may have the relation

$$\widetilde{m}\,[0,\ \lambda a] = \lambda \widetilde{m}\,[0,\ a]\,.$$

Thus, the situation that points make up the occupation of a set in a space $\boldsymbol{E}$, must induce a spatial relation of each point to the space $\boldsymbol{E}$, which admits a quantitative character toward a point. By this reason we associate a point P with an infinitesimal piece of space $((P))$ supposed to be occupied by P, and call $((P))$ the (*point*) *occupation* of P in respect to the a priori measure $\widetilde{m}$. It will then be considerable that e.g.

$$((x)) = (x-0,\ x+0),\ [x,\ x+0),\ \left[x-\frac{1}{3}0,\ x+\frac{2}{3}0\right)\ \text{etc.}.$$

Since a set is defined as an aggregate of points, the number of the points contained in it may naturally be abstracted. Namely, we define $\tilde{m}M$ in the form

$$\tilde{m}M = \mu\mathfrak{n}(M) \tag{1. 1}$$

where $$\mu = \tilde{m}((P))\ (P \in M)$$

and $\mathfrak{n}(M)$ is called the *inversion number* of M.

In the above case, the size of a point P in $\boldsymbol{E}$ is considered to be everywhere equal; the measure $\tilde{m}$ is then called a *normal* (a priori) measure. When

$$\mu_P = \tilde{m}((P))$$

is not everywhere equal, $\tilde{m}$ is said to be *abnormal*. The integral construction of $\tilde{m}M$ is given by

$$\tilde{m}M = \underset{P \in M}{\mathfrak{S}} \mu_P \tag{1. 2}$$

which may coincide with the classical formula

$$\tilde{m}M = \int dP.$$

μ_P is called the (*point*) *dimension* (or the $\tilde{m}$-*dimension*) of P. The sum of all the point occupations of A is called the (*total*) *occupation* of A, which will give a concrete concept, equivalent to that of a set, to comprehend the spatial construction of the integration (1. 2).

In case of n dimensions, a point P being represented by the cartesian coordinate $(x_1, \cdots, x_n)$, the point dimension of P is given in the form

$$\mu_P = \mu_{x_1} \cdots \mu_{x_n},$$

where μ_{x_k} is regarded as the projection of μ_P on the k-th axis. Then, μ_P shall naturally correspond to the integral element

$$dx_1 \cdots dx_n$$

in the classical theory of integral.

The notion of the size of a point may give a convenient medium of illustration. For instance, in the plane geometry, if the point P_k is represented by the polar coordinate (ρ_k, θ_k) $(k=1, 2)$, we have

$$\mu_{P_1}/\mu_{P_2} = \rho_1\mu_{\rho_1}\mu_{\theta_1}/\rho_2\mu_{\rho_2}\mu_{\theta_2} = \rho_1/\rho_2,$$

if μ_ρ and μ_θ are given as normal dimensions. Then, the ratio of the sizes of P_1 and P_2 shall be regarded as equal to ρ_1/ρ_2.

In case of an abnormal (a priori) measure $\tilde{m}$, the inversion number $\mathfrak{n}(M)$ of a set M cannot be given by (1. 1). In this case, the following formulation may give a help. If

$$\lambda(P) = \mu_P/\mu_Q$$

Q being a fixed point, we shall have

$$\bar{\lambda} = \widetilde{m}M/\mu_Q \mathfrak{n}(M)$$

$\bar{\lambda}$ being the mean value of $\bar{\lambda}(P)$ for $P \in M$.

2. Resilience

The representative convention such that

$$1 = 0.99\cdots, \; 0.23 = 0.2299\cdots \text{etc.}$$

may be said very convenient in point that any real number can, through this modification, be uniquely expressed; still, the statement that the limit of the values

$$0.9, \; 0.99, \cdots$$

is equal to 1, may not always be cosidered as strictly appropriate. If exactly, it must be that

$$0.99\cdots = 1-0\,.$$

In effect, if a univoque function $f(x)$ is discontinuous on the left hand of a point x, then it must be that

$$f(x) \neq f(x-0)\,.$$

It may generally be admitted that, in the space of real numbers, any point x has no point just prior or just posterior to it. This situation may be considered coincident with the fact that two intervals of different length can be set in one-one correspondence of points. However, if these intervals be restricted to the same normal measure, one-one corresponcence must only mean an equal measure of length. Under the normal measure system, (0, 100) is regarded to contain 100 times as many points as (0, 1). That in such ways as above-stated, points are distributed to sets, shall be illustrated as points occupy their positions in some repelling state each other. We abstract the notion of this repelling tendency to be associated with each individual point P and call it the *resilience* of P. Then μ_P may be thought as the measure of a sort of total resilience around P. In case of 2 dimensions, a point (x, y) is considered to have resiliences in positive and negative directions along x- and y-axes. If ABC is a triangle and if any point of the side BC has two resiliences, one parallel to BA and one prallel to CA, then the total linear measure of the resiliences on BC may be counted as AB+AC. Thus, the well-known paradoxical assertion that the length of BC must be equal to AB+AC, may actually be turned to be reasonable.

3. Probabilism

In the classical theory of sets, if 'a set A' is merely supposed to be existent, without any practical confirmation such as is seen in cases of a rectangle, sphere etc., it may not give any real fact and may not be other than a nonsensical designation, even when it is provided with the condition $\widetilde{m}A=1$. This is because the general notion of a set is not positively construed with measure theoretical foundations.

Now, by the following table, let us compare the definition of the normal a priori measure with that of the notion of a descriptive set* of points:

($\mathbf{M}_1$) Any point P of the space $\boldsymbol{E}$ maintains the same size of occupation $(\!(P)\!)$ and $\widetilde{m}(\!(P)\!) = \mu$;	(S_1) Each point of $\boldsymbol{E}$ has its own position and can be distinguished from other points;
($\mathbf{M}_2$) The total occupation of the points of A makes up $\widetilde{m}A$ satisfying the formula $$\widetilde{m}A = \mathfrak{n}(A)\mu.$$	(S_2) That A is an aggregate of points in $\boldsymbol{E}$ is confirmable by means of the criterion $$(\forall P \in \boldsymbol{E})\ (P \in A \vee P \notin A).$$

The total occupation of a set A may naturally be compared to the state that A is filled with some substance. In effect, the space $\boldsymbol{E}$, in physics, is usually considered to be everywhere filled with 'ether'. Then μ shall mean the mass-value of the ether equally assigned to each point and $\widetilde{m}A$ the total mass of the ether distributed to A.

As to (S_1), that a point is distinguished from other points, shall, in the physical sense, mean that P is distinguished in the relation to the circumstance that an aggregate of points directly causes the total sum of the ether to be distributed to it. Such a physical distinction may not evidently be attained but for the notion of 'density' of the ether of A in any neighborhood of the point P. Besides, the density of the ether of A may directly be interpreted as the probability of occurrence of the points of A in a neighborhood of P. Thus, we may expound it: that a set A is determined as an aggregate of points in $\boldsymbol{E}$, must coincide with the fact that, in any sphere S we have

$$\widetilde{m}A \cap S / \widetilde{m}S = \Pr(P \in A) \tag{3. 1}$$

P being an aleatory variable point restricted within S. We adopt (3. 1) as the probabilistic definition of $\widetilde{m}A$ in relation to $\widetilde{m}S$. $\widetilde{m}S$ is of a trivial measurability. When $A \subseteq S$, $\widetilde{m}A = \Pr(P \in A)\,\widetilde{m}S$.

On the above-stated foundation, it is remarkably important that any (descriptive) set must be $\widetilde{m}$-measurable. This is apparently the effect of the physical interpretation of the space $\boldsymbol{E}$ by means of 'ether'. If we could pour the ether distributed to a set A into a vessel and weigh it, the mass-value $\widetilde{m}A$ might surely be obtained. With respect to (3. 1) we see that, probabilism, in this case, plays a role to turn the microscopic sight of a point occupation toward the macroscopic one of the total occupation of a set. As for the inversion number, the following formula holds:

$$\Pr(P \in A\,(P \in M \ \&\ A \subseteq M)) = \mathfrak{n}(A)/\mathfrak{n}(M)$$

on condition that $\mathfrak{n}$ is the inversion number provided for a normal measure.

If $\widetilde{m}$ is a normal a priori measure, and if we have

* An aggregate of points satisfying the conditions of (S_2) is a *descriptive* set.

$$(\forall A \subseteq \boldsymbol{E})\ (\tilde{m}A = \tilde{m}_1 A)$$

we call $\tilde{m}_1$ an a priori measure too, even when $\tilde{m}_1((P))$ is not everywhere equal. If

$$\tilde{m}_1((P)) \neq \tilde{m}_1((Q)),$$

it simply means that the size of $((P))$ is not equal to that of $((Q))$. Therefore, that $\tilde{m}$ is normal means that all of $((P))$ are taken to be of equal size. If

$$\tilde{m}_1((P))/\tilde{m}_1((Q)) > 1,$$

the probability of occurrence of the point P is naturally larger than that of the point Q. Since the construction of $\boldsymbol{E}$ i. r. t.* $\tilde{m}_1$ thus differs from that i. r. t. a normal measure $\tilde{m}$, the inversion number of a set i. r. t. $\tilde{m}_1$ must also differ from that i. r. t. $\tilde{m}$. Denoting the inversion number of a set A i. r. t. $\tilde{m}_1$ by $\mathfrak{n}(A, \tilde{m}_1)$, we have

$$\bar{\mu}_1/\mu = \mathfrak{n}(A,\ \tilde{m})/\mathfrak{n}(A,\ \tilde{m}_1),$$

where $\bar{\mu}_1$ is the mean $\tilde{m}_1$-dimension (i. e. the mean of $\tilde{m}_1((P))$ for P) in A and μ is the normal $\tilde{m}$-dimension.

4. Complete Additivity

If the family of sets $(M_\iota)(\iota \in I)$, I being a set of ordinal numbers, satisfies the condition

$$(\forall \iota \in I)\ (M_\iota \subseteq K)$$

and if
$$0 \leqslant \tilde{m}K < \infty,$$

then (M_ι) is said to be *$\tilde{m}$-bounded.* In this section, we suppose that (M_ι) is $\tilde{m}$-bounded and monotone increasing viz.

$$(\forall \iota,\ \kappa \in I)\ (\iota < \kappa \Rightarrow M_\iota \subseteq M_\kappa),$$

and
$$M = \cup M_\iota. \tag{4. 1}$$

(4. 1) naturally suggests that M is the limiting set of (M_ι). Besides, since M is, in our view, considered $\tilde{m}$-measurable without exception, it shall be defined that M is the limiting set of a $\tilde{m}$-bounded monotone increasing family of sets (M_ι) when and only when

$$\cap (M - M_\iota) = \text{void}\ \&\ \inf\ \tilde{m}(M - M_\iota) = 0.$$

Since the set of values $\tilde{m}M_\iota(\iota \in I)$ is, by supposition, a bounded set of real numbers, there exists a sequence $(M_{\iota_k})(k = 1, 2, \cdots)$ such that

$$\lim\ \tilde{m}M_{\iota_k} = c = \sup\ \tilde{m}M_\iota.$$

Then, in empiricism, it is easily verified that

$$c = \tilde{m}M.$$

Thus $\tilde{m}$ is found to be a completely additive set function.

* 'i. r. t.' and 'w. r. t.' are rendered 'in respect to' and 'with respect to' respectively.

In empiricism, a limiting object is admitted when and only when it can be approached by an enumerable process of stepping. So, in the case above-mentioned, it must be that

$$(\exists \iota_k (k=1, 2, \cdots)) \ (\cup M_{\iota_k} = M).$$

Still, it is notable that there is an additive set function in $\boldsymbol{E}$, which is not completely additive, even when all points are given equal assignment by it. Such a function is called an *ultra set function*[1].

5. Application

In order to construct an a priori measure we assumed spatial point occupations $((P))$, which precisely fill up the whole space $\boldsymbol{E}$ without overlapping. By this way of construction, if the system of $((P))(P \in \boldsymbol{E})$ is given, the corresponding a priori measure $\tilde{m}$ is completely determined and vice versa. In this regard, $((P))$ is called the *$\tilde{m}$-occupation* of the point P. Now, let us assume that a mass quantity γ_P is univoquly assigned to each $\tilde{m}$-occupation $((P))$ to define a set function $\tilde{\gamma}(M)$ in the form

$$\tilde{\gamma}M(=\tilde{\gamma}(M)) = \underset{P \in M}{\mathfrak{S}} \gamma_P , \qquad (5.\ 1)$$

which means that the quantities γ_P are summed up through the total occupation of a set M. $\tilde{\gamma}$ is called an *application* and $\tilde{m}$ is then called the *carrier* of $\tilde{\gamma}$ in the meaning that the spatial construction for the integral (5. 1) is given by the system of $\tilde{m}$-occupations $((P))$. Then, it is naturally assumed that

$$\gamma_P = \tilde{\gamma}((P)).$$

γ_P is called the *point appication* of P w. r. t. $\tilde{\gamma}$. When γ_P is infinitesimal, we write

$$\gamma_P = \triangle\!\!\!\!\bigcirc ;$$

when non-negative and infinitesimal

$$\circledcirc \leqslant \gamma_P \leqslant \triangle\!\!\!\!\bigcirc .$$

$\circledcirc$ indicates 'empty null' which means the vacancy of quantity. In this section, we confine our argument to the case of non-negative and bounded $\tilde{\gamma}$. Then, it may easily be seen that values of γ_P must be at most infinitesimal except at most an enumerable number of them. A general application may be expressed as a difference of two non-negative ones.

If we could pour all of γ_P distributed to A together into a vessel and weigh them, the value $\tilde{\gamma}A$ might surely be obtained. If constructively, partitions of a set A may be brought forward to be observed along with $\tilde{\gamma}$. However, in empiricism, an observable partition must be limited to an enumerable one. Thus, we are forced to have the definition as follows:

Definition. *If, for any enumerable partition* $(M_k)(k=1, 2, \cdots)$ *of a set* M, *we have*

$$\tilde{\gamma}M = \sum_{k=1}^{\infty} \tilde{\gamma}M_k ,$$

then $\tilde{\gamma}M$ *is represented in the form*

$$\tilde{\gamma}M = \mathfrak{S}\gamma_P$$

with $$\gamma_P = \tilde{\gamma}((P)),$$

$((P))$ *being point occupations i. r. t. a certain a priori measure.*

In fine, $\tilde{\gamma}$ is defined by (5. 1) as a completely additive set function. This may be thought as a merit of empiricism. As for the quantitative criticism on γ_P, we may sort out the following four cases: (i) $\gamma_P = \circledcirc$; (ii) $0 < \gamma_P < \infty$; (iii) $0 < f(P) \equiv \gamma_P / \mu_P < \infty$; (iv) $\circledcirc < \gamma_P \leqslant \triangle$ & $f(P) = 0 \vee \infty$, μ_P being the point dimension of the carrier $\tilde{m}$. The complement of the set $\{P: \gamma_P = \circledcirc\}$ is the support of $\tilde{\gamma}$. In the part of (iii), $\tilde{\gamma}$ may be expressed as an integral

$$\mathfrak{S} f(P) \mu_P \text{ or } \int f(P) dP.$$

We assume the case (iv) to be possible, but do not make any detailed explanation on it here[2].

An additive set function f (in $\boldsymbol{E}$) which is neither an a priori measure nor an application, is an ultra set function. In this case, the only formula generally promised for f is that

$$(\forall A, B \subseteq \boldsymbol{E}) \ (f(A \cup B) = f(A) + f(B) - f(A \cap B)).$$

II. Logic and Empiricism

1. Ranging

If a chain or a concatenation of symbols or words is certainly read as indicating or designating some objects or some state of the objects, it is called a *description* (*in the generalized sense*). When exclusive cases for certain situations are taken as elements, the set

$$R(A) = \{\xi : A \text{ is true in } \xi\}$$

is called the *usual deductive range* of the description A. Then, implication '$\Rightarrow$' may be defined by

$$A \Rightarrow B. \equiv .R(A) \subseteq R(B) \tag{1. 1}$$

on condition that $R(A) \neq$ void. More generally, we assume that to any description A (of a given family of descriptions) uniquely corresponds a set $R(A)$ (of elements of a given universe); then, by the implication defined by (1. 1), we will obtain a deductive system of logical language. If $\boldsymbol{U}$ is a universe of objects and

$$R(A) \subseteq \boldsymbol{U} \ \& \ R(A) \neq \text{void},$$

then A is called a *description* (*standing*) *on* $\boldsymbol{U}$. For a family of desriptions $\mathfrak{A}$ it may not always be possible to find a universe $\boldsymbol{U}$ such that

$$\bigcup_{A \in \mathfrak{A}} R(A) \subseteq \boldsymbol{U}. \tag{1. 2}$$

If $\boldsymbol{U}$ is existent and satisfies (1. 2), we say that $A(\in\mathfrak{A})$ or $\mathfrak{A}$ is given a *ranging* in $\boldsymbol{U}$, and then call $\boldsymbol{U}$ the *range universe* of this ranging.

A course of logic usually involves a definition of the level to correspond to a predicate or an object. If $\boldsymbol{U}$ is a range universe of which all elements are descriptions, then the elements of $\boldsymbol{U}$ shall be regarded as of the same level. If $\boldsymbol{U}_1$ and $\boldsymbol{U}_2$ are range universes and if any element of $\boldsymbol{U}_2$ is either a description on $\boldsymbol{U}_1$ or a relation between subsets of $\boldsymbol{U}_1$, then $\boldsymbol{U}_2$ is said to be of *higher level* than $\boldsymbol{U}_1$, in that any element of $\boldsymbol{U}_2$ is regarded as of higher level than any element of $\boldsymbol{U}_1$. If $\boldsymbol{U}_0 \neq$ void and there is no universe to be of lower level than $\boldsymbol{U}_0$, the level of $\boldsymbol{U}_0$ is zero. However, it appears that essentially the levels of objects are determined relatively and not absolutely. For instance: when a line is defined by a pair of points, the line will be thought to be of higher level than the points; but, when a point is defined by a pair of lines, the point will be of higher level. Such being the conditions, we will take the notion of the level only to be sometimes conveniently used in the relative meaning. Descriptions on the same universe $\boldsymbol{U}$ are of the same level, because their ranges then are equally subsets of $\boldsymbol{U}$.

If descriptions A and B are of the same level, following 8 cases are distinguished:

α_1: $R(A) =$ void & $R(B) =$ void; $\quad\alpha_2$: $R(A) =$ void & $R(B) \neq$ void;
α_3: $R(A) \neq$ void & $R(B) =$ void; $\quad\alpha_4$: $R(A) \neq$ void & $R(A) \subset R(B)$;
α_5: $R(B) \neq$ void & $R(B) \subset R(A)$; $\quad\alpha_6$: $R(A) \neq$ void & $R(A) = R(B)$;
α_7: $R(A) - R(B) \neq$ void & $R(B) - R(A) \neq$ void & $R(A) \cap R(B) \neq$ void;
α_8: $R(A) \neq$ void & $R(B) \neq$ void & $R(A) \cap R(B) =$ void.

Then, taking $\boldsymbol{U} = \{\alpha_1, \alpha_2, \cdots, \alpha_8\}$ as the universe, we may have

$$R(A \Rightarrow B) = \{\alpha_1, \alpha_2, \alpha_4, \alpha_6\}.$$

'$\vdash A$' is usually rendered 'A is true'. However, in this paper, we let '$\vdash A$' mean 'A is possible' (i. e. 'A is not impossible'). '$\sim\vdash A$' is the negation of '$\vdash A$' and is rendered 'A is impossible' or 'A is false'. 'A' itself cannot be rendered as a description on $\boldsymbol{U}$, whereas $\vdash A$ and $\sim\vdash A$ stand on $\boldsymbol{U}$. In effect, we still have

$$\left.\begin{aligned} R(\vdash A) &= \{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}, \\ R(\vdash B) &= \{\alpha_2, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}, \\ R(\sim\vdash A) = \{\alpha_1, \alpha_2\} \text{ and } R(\sim\vdash B) &= \{\alpha_1, \alpha_3\}. \end{aligned}\right\} \tag{1. 3}$$

In fine, $(A \Rightarrow B) \wedge A$ is not a description on $\boldsymbol{U}$, but $(A \Rightarrow B) \wedge (\vdash A)$ and $(A \Rightarrow B) \wedge (\sim\vdash A)$ are ones on $\boldsymbol{U}$.

Now, since

$$R((A \Rightarrow B) \wedge (\sim\vdash A)) = R(A \Rightarrow B) \cap R(\sim\vdash A) = \{\alpha_1, \alpha_2\},$$

with respect to (1. 3) we have

$$R((A \Rightarrow B) \wedge (\sim \vdash A)) \not\subseteq R(\vdash B),\ \not\subseteq R(\sim \vdash B),\ \text{but } \subseteq R(\vdash B) \cup R(\sim \vdash B).$$

Hence we conclude

$$(A \Rightarrow B) \wedge (\sim \vdash A) \not\Rightarrow \vdash B,\ \not\Rightarrow \sim \vdash B,\ \text{but } \Rightarrow \vdash (B \vee \sim B). \tag{1. 4}$$

It is remarkable that the result (1. 4) is incompatible with the assertion 'fallacy implies any event', which is professed by some sect of symbolic logicians.

2. Event Complex

A description shall, in itself, be regarded as an event. If its usual range $\neq$ void, it is called a *possible event* and if $=$void an *impossible* one. Though the terms 'event', 'possible' and 'impossible' are, originally, of the theory of probability, they are rather more lucid than the corresponding terms of pure logic and may even be preferable in point of straightness for the empiricist view. With this terminology, we may straightly pass to the statistical view if needed.

If the premises, notions or relations among them, and the available referential facts in the context of a theme are resolved into a finite number of descriptions $\boldsymbol{A} \equiv \{A_1, \cdots, A_n\}$ of which all are regarded as of the same level, then the state construction defined in the form

$$\mathfrak{c}(A) \equiv \vee_{k=1}^{n} (A_k \vee \sim A_k)$$

is called the *event complex* (or simply the *complex*) *generated by* $\boldsymbol{A}$. In this case, partial products of 2n events A_k, $\sim A_k$ $(k=1, \cdots, n)$, which do not vanish, make, in all, a finite set

$$\Gamma(\boldsymbol{A}) = (\Gamma_j)(j=1, \cdots, \nu),$$

and Γ_j are found to be mutually exclusive events. $\Gamma(\boldsymbol{A})$ is called *$\boldsymbol{A}$-aspect* of the theme.

If we take $\Gamma(\boldsymbol{A})$ as the range universe, we may sufficiently transact inferences on the theme by means of the language standing on $\Gamma(\boldsymbol{A})$, i. e. the language which has $\Gamma(\boldsymbol{A})$ as the universe of individuals.

3. Inductive Range

Induction too is proceeded on contradistinction of some implicative relations. So then, a ranging must thereupon be contrived to define the implication. Deductive ranges are found incompatible with this purpose. The deductive range of a description A comprises possible events of A as its elements, because, in a deductive case, the point of observation is whether the object is possible (or true) or not. However, in an inductive case, observation rests only on the residual part of inspection, so the ranging should also be defined on this part.

Assuming that $\mathfrak{P}$ is a set of propositions and is provided with a criterion φ which is tested on subsets of $\mathfrak{P}$, if a subset $\boldsymbol{P}$ of $\mathfrak{P}$ conforms to φ, we write $\varphi \vdash \boldsymbol{P}$, and if not, $\sim \varphi \vdash \boldsymbol{P}$. In addition, we assume that φ satisfies the following two properties :

$$\textit{descriptiveness}:\quad (\forall \boldsymbol{P}\subseteq \mathfrak{P})\ (\varphi\vdash\boldsymbol{P}\vee\sim\varphi\vdash\boldsymbol{P});$$
$$\textit{regressiveness}:\quad \boldsymbol{P}\subseteq\boldsymbol{Q}\subseteq\mathfrak{P}\ \&\ \varphi\vdash\boldsymbol{Q}.\Rightarrow.\varphi\vdash\boldsymbol{P}. \tag{3. 1}$$

In this case, we define a rang $R(\boldsymbol{P})$ by the stipulation that

$$R(\boldsymbol{P})=\mathfrak{P}-\boldsymbol{P}\ \text{when}\ \varphi\vdash\boldsymbol{P} \tag{3. 2}$$

and
$$=\text{void when}\ \sim\varphi\vdash\boldsymbol{P}.$$

Then, the implication appearing in (3. 1) may be realized by the definition:

$$\boldsymbol{P}\Rightarrow\boldsymbol{Q}.\equiv.R(\boldsymbol{P})\subseteq R(\boldsymbol{Q}) \tag{3. 3}$$

on condition that $R(\boldsymbol{P})\neq$void. It will be needless to say that the left side of (3. 3) just means $\varphi\vdash\boldsymbol{P}\Rightarrow\varphi\vdash\boldsymbol{Q}$. The range defined by (3. 2) is called an *inductive range.*

By means of the principle of cut approach[3] in empiricism, we may directly attain the theorem:

Proposition 3. 1. *If* $\mathfrak{P}$ *is a set of propositions with a descriptive and regressive criterion* φ *to be tested on its subsets, and if*

$$\sim\varphi\vdash\mathfrak{P}$$

and
$$(\exists\boldsymbol{P}\subset\mathfrak{P})\ (\boldsymbol{P}\neq \textit{void and}\ \varphi\vdash\boldsymbol{P}),$$

then there are two sequences of subsets of $\mathfrak{P}$ $(\boldsymbol{P}_k)$ and $(\boldsymbol{Q}_k)$ $(k=1, 2, \cdots)$ *such that*:

(i) $\boldsymbol{P}_1\subseteq\boldsymbol{P}_2\subseteq\cdots\subseteq\boldsymbol{Q}_2\subseteq\boldsymbol{Q}_1\subseteq\mathfrak{P}$;

(ii) $\cup\boldsymbol{P}_k=\cap\boldsymbol{Q}_k$;

(iii) $(\forall k)\ (\varphi\vdash\boldsymbol{P}_k\ \&\ \sim\varphi\vdash\boldsymbol{Q}_k)$.

4. Unmaximizable Case

If we apply the principle of transfinite induction, Proposition 3. 1 may be altered to the following result:

[T]. *Under the same conditions assumed in Proposition* 3. 1, *there exists a family of subsets of* $\mathfrak{P}$ $(\boldsymbol{P}_\lambda)$ $(\lambda\in\Lambda)$ *with an indication set* Λ *of ordinal numbers such that*:

(i) $(\forall\lambda,\ \mu\in\Lambda)\ (\lambda<\mu.\Rightarrow.\boldsymbol{P}_\lambda\subseteq\boldsymbol{P}_\mu)^*$;

(ii) $(\forall\lambda\in\Lambda)\ (\varphi\vdash\boldsymbol{P}_\lambda)$;

(iii) $\boldsymbol{Q}\subseteq\mathfrak{P}\ \&\ \boldsymbol{Q}\supset\widetilde{\boldsymbol{P}}=\cup\boldsymbol{P}_\lambda.\Rightarrow.\sim\varphi\vdash\boldsymbol{Q}$.

$\widetilde{\boldsymbol{P}}$ appearing in (iii) may be regarded as a supremum w. r. t. φ. When such $\widetilde{\boldsymbol{P}}$ exists, φ is said to be *maximizable on* $\mathfrak{P}$. [T] itself, however, is denied in empiricism, by the following example.

We may take a euclidian space (of finite dimension) $\boldsymbol{E}$ as $\mathfrak{P}$ in the sense that a point 'P' is also regarded as a symbol 'P' rendered '$P\in\boldsymbol{E}$'. φ be defined by

* '$\Rightarrow$' shall henceforth be read as' ,then we have'. Such it may be read in either case of a deductive or an inductive range.

$$\varphi \vdash A. \equiv .\tilde{m}A \leqslant c, \tag{4. 1}$$

c being a fixed finite positive number. If [T] in this case holds, there is a family of subsets of $\boldsymbol{E}$ $\mathfrak{A}=(A_\lambda)(\lambda \in \Lambda)$ such that

$$\lambda < \mu. \Rightarrow .A_\lambda \subseteq A_\mu$$

and if

$$\tilde{A} = \cup A_\lambda$$

we may have

$$(\forall B \subseteq \boldsymbol{E})\ (B \supset \tilde{A}. \Rightarrow .\tilde{m}B > c).$$

Since $\tilde{m}$ is an a priori measure in $\boldsymbol{E}$, we then have

$$\tilde{m}\tilde{A} = \sup(\tilde{m}A_\lambda) \tag{4. 2}$$

so that

$$\tilde{m}\tilde{A} = c.$$

Therefore, if we take an enumerable set N in $\boldsymbol{E}-A$ and define B as

$$B = A \cup N$$

we may directly have

$$B \supset \tilde{A}\ \&\ \tilde{m}B = c.$$

Thus φ defined by (4. 1) cannot be maximizable. It is remarkable that the above-shown contradiction (to the existence of $\tilde{A}$) is concluded only by the characteristic relation (4. 2) of an a priori meaure $\tilde{m}$ and not by any restriction on ordinal numbers. If we mean to insist [T], we must then necessarily renounce the property (4. 2) of $\tilde{m}$ and thereafter assert either $\tilde{A}$ to be denied its $\tilde{m}$-measurabilty or $\tilde{m}$ itself to be denied its complete additivity.

Since we shall be resting on the theory of a priori measure, we may not renounce (4. 2). Thus, we encounter an unexpected obstruction to the principle of trans-induction which was attempted to be an alternative renovation of the principle of transfinite induction. It is very regretful that here the preseut author must change his previous announcement that the principle of trans-induction may be made well-established by means of the empiricist principle of cut approach[2]. Some reflection will show us that such an unmaximizable case as above discussed, may appear only when the residual part for inspection with respect to $\sim\varphi$ dose not vanish out. So then, it is considered relevant to restrict the conditions as follows.

If φ is a regressive criterion on subsets of $\mathfrak{P}$ and if

$$(\forall \boldsymbol{P} \subseteq \mathfrak{P})\ (\varphi \vdash \boldsymbol{P}\ \&\ \boldsymbol{P}^c \neq \text{void}. \Rightarrow (\exists \boldsymbol{Q} \subseteq \mathfrak{P})(\boldsymbol{P} \subset \boldsymbol{Q}\ \&\ \varphi \vdash \boldsymbol{Q}))^*,$$

then φ is said to be *insuppressible* on $\mathfrak{P}$. Then, it is easily shown that φ is insuppressible whenever φ is unmaximizable on $\mathfrak{P}$. We now assume an operator Φ called a *φ-inspector* being defined as follows:

$$\text{(i)}\quad \boldsymbol{P} \subseteq \Phi(\boldsymbol{P}); \qquad \text{(ii)}\quad \boldsymbol{P} \subset \boldsymbol{Q}. \Rightarrow .\Phi(\boldsymbol{P}) \subseteq \Phi(\boldsymbol{Q});$$

* $\boldsymbol{P}^c \equiv \mathfrak{P} - \boldsymbol{P}$ and $\Phi(\boldsymbol{P})^c \equiv \mathfrak{P} - \Phi(\boldsymbol{P})$.

(iii) $\Phi(\boldsymbol{P}) \neq \Phi(\boldsymbol{Q}). \Rightarrow .\boldsymbol{P} \neq \boldsymbol{Q}$;

(iv) $\varphi \vdash \boldsymbol{P}$ & $\boldsymbol{P} \subseteq \boldsymbol{Q}$ & $\Phi(\boldsymbol{Q})^c \neq$ void. $\Rightarrow .(\exists \boldsymbol{R} \subseteq \mathfrak{P})$ $(\boldsymbol{P} \subset \boldsymbol{R}$ & $\varphi \vdash \boldsymbol{R}$ & $\Phi(\boldsymbol{Q}) \subseteq \Phi(\boldsymbol{R}))$.

In this case, the set $\boldsymbol{P}$ which holds $\varphi \vdash \boldsymbol{P}$ will be enlarged unless $\Phi(\boldsymbol{P})$ vanishes. So we may have:

Proposition 4. 1. *Under the same designations with* [T], *if φ accompanies a φ-inspector Φ, we may have*

$$\varphi \vdash \widetilde{\boldsymbol{P}}$$

only when

$$\Phi(\widetilde{\boldsymbol{P}}) = \mathfrak{P}.$$

Besides, the principle of trans-induction shall be introduced in the renovated form as follows:

Principle of Trans-induction. *If φ is a descriptive and regressive criterion on $\mathfrak{P}$ and is provided with a φ-inspector Φ, then there is a monotone increasing sequence of subsets of $\mathfrak{P}$ $(\boldsymbol{P}_k)$ $(k=1, 2, \cdots)$ such that*

$$(\forall k)\ (\varphi \vdash \boldsymbol{P}_k)$$

and

$$\cap \Phi(\boldsymbol{P}_k)^c = \textit{void}.$$

This principle shall, of course, rest on the ground of empiricism, i. e. on the view that any limiting process can be realized by an enumerable stepping whenever it is found possible. As for the limiting set $\widetilde{\boldsymbol{P}}$ of the sequence, whether $\varphi \vdash \widetilde{\boldsymbol{P}}$ or $\sim\varphi \vdash \widetilde{\boldsymbol{P}}$ cannot generally be presented in advance.

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